Cycloids and Brachistochrones

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Lab 1

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A straight line is indeed the shortest distance between two points. However, it is not necessarily the path that consumes the smallest amount of travel time for a particle. Given the condition that a particle exists in a field of uniform force (constant in both magnitude and direction), and that the force's direction is tangent (or nearly so) to the endpoint of a cycloid-shaped path, then if the particle follows this path from point A to point B (where point A has a higher potential energy due to the force involved) then the particle will arrive at point B sooner than it would have had it followed a straight line path to point B. This shall be proved in the following paragraphs. It will not be proved, however, that the cycloid is the curve that will have the shortest transit time out of all curves that exit. This requires math that is unknown to the author. Integration and parameterization will be used to this end in conjunction with the Mathematical Visualization Toolkit applet from the Applied Mathematics Department at the University of Colorado, Boulder located at:

http://amath-www.colorado.edu/java/MVT/packages/MathTool.html.

A cycloid is the curve that is traced in space by a fixed point on the perimeter of a circle that is rolling along a flat surface. Figure 1 shows this graphically. A cycloid is commonly parameterized by the following equations:

$$x(w)=x(0)+(a/g)(w-\sin(w))$$

$$y(w)=y(0)-(a/g)(1-\cos(w))$$

$$0<w<wf$$

Here, ‘g’ is the acceleration due to gravity (9.81 m/sec^2), ’w’ is a parameter, and the

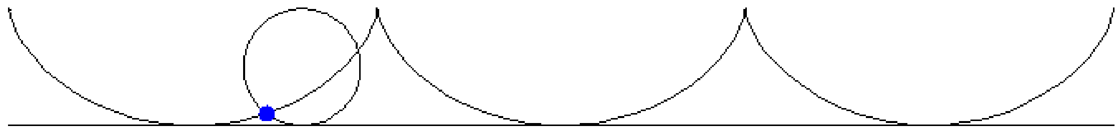

FIGURE 1

ratio ‘a/g’ that appears in the parameterization is nothing more than the radius of the rolling circle.

The path that will be taken by the test particle will be a portion of a cycloid. The

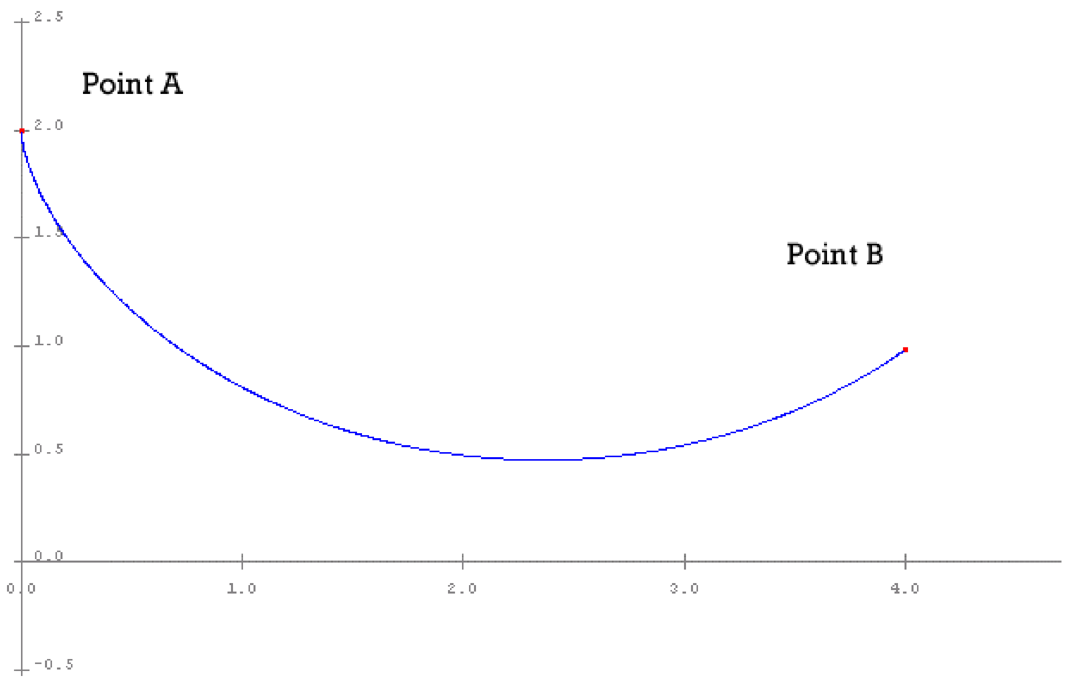

FIGURE 2

motion will be from left to right, top to bottom. The constant force applied will be due to gravity, which will act vertically, and parallel with the edges of this sheet of paper. Figure 2 shows the test setup. The starting coordinate is A(0, 2), and the ending coordinate is B(4, 1). The test particle will follow the cycloid as shown. The time taken to travel in a straight line from point A to point B will then be compared.

In order that the line work with the methodology, it cannot be in the standard y = mx + b form. It must be parameterized. In order to do this, a vector of the line is formed. See Appendix A for the details of this operation. The final parameterizations of the line are as follows:

$$\mathbf{x = 4*w/w}f$$

$$\mathbf{y = 2 - w/w}f$$

With these parameter equations, the following equations can be used for both the cycloid and the line.

The length of this cycloid segment can be computed by using the arc length integral. It is as follows:

$$L = \int ds = \int (x'(w)^2+y'(w)^2)^{1/2}\, dw$$

Here, the second integral is taken from w0 to wf. See Appendix B for the details of this calculation. Calculating this integral yields a cycloid arc length of 4.748 meters. Standard intuition allows the common person to realize that the length of the line that lies between point A and point B is shorter in length than that of the cycloid. It can be computed by using Pythagorean's Theorem. Since the coordinates of the two points are known, the line's length can be calculated as follows:

$$L = ((xB-xA)^2+(yB-yA)^2)^{1/2}$$

The length of the line computed is the square root of 17 meters, which is approximately equal to 4.123 meters. This is definitely shorter in length than the cycloid arc is.

A standard result from classical mechanics is that the time it takes for a particle to move along a curve parameterized by (x(w),y(w)) is given by:

$$T = \int ds/v = \int (1/v)*(x'(w)^2+y'(w)^2)^{1/2}\ dw$$

where v is the velocity that the test particle travels at each point along the curve. The second integral again is taken from w0 to wf. In cases where the only force acting on the particle is a constant gravitational force, a standard conservation of energy argument will tell you that the velocity is given by:

$$v = (2g(y(0)-y(w)))^{1/2}$$

The computation details for these equations can be found in Appendix D. The results of the time integral calculation show the time taken for the test particle on the cycloid path to be approximately 1.21 seconds, while the time taken on the line path is approximately 1.86 seconds. This shows that the time taken along a cycloidal path is definitely shorter under the given conditions.

In conclusion, with the above computations, it has been shown that a particle will take less time to travel between a given point A and point B while in a constant force field if it follows the path of a cycloid instead of a line, in spite of the fact that the line's length is shorter than the length of the cycloid arc. Further research could be conducted in this area to see at what maximum angle the force's direction could be at while still having a faster cycloid path. But, this will be left to other, greater mathematicians.